Fiber products (Har II 3, Shaf V 4.1)

Let S be a scheme. A <u>scheme over S</u>, or an <u>S-scheme</u> is a scheme X together with a morphism $X \to S$. A <u>morphism of S-schemes</u> is a morphism $X \to Y$ such that this diagram commutes:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

If A is a ring, $(\operatorname{Spec} A)$-schemes are often called A-schemes.

If X and Y are two S-schemes, we define the <u>fiber product over S</u>, denoted $X \times_S Y$ to be a scheme w/ morphisms $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$ compatible w/ maps $X \to S$, $Y \to S$ (i.e. orange + green diagram commutes) such that if Z is another S-scheme, w/ maps $f : Z \to X$, $g : Z \to Y$ such that the purple + green diagram commutes, then there is a unique morphism $\theta : Z \to X \times_S Y$ such that the purple, yellow, + orange diagrams commute.

$$\begin{tikzcd} Z \arrow[dr, dashed, "\theta"] \arrow[drr, "f"] \arrow[ddr, "g"'] & & \\ & X \times_S Y \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d] \\ & Y \arrow[r] & S \end{tikzcd}$$

Caution: This is <u>not</u> a product on the underlying topological spaces!

Note that if the fiber product exists, then it must be unique up to isomorphism:

If W and Z are both fiber products:

$$\begin{tikzcd} W \arrow[dr, shift left] \arrow[rr, bend left] \arrow[ddr, bend right] & & \\ & Z \arrow[ul, shift left] \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & S \end{tikzcd}$$

(convince yourself of this if you've never seen this argument before!)

We show existence first in affine case:

Prop: If $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and $S = \operatorname{Spec} R$ s.t. A and B are R-algebras, then $\operatorname{Spec}(A \otimes_R B)$ is the fiber product $X \times_S Y$.

Pf: We have the following diagram of rings:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & A \arrow[d] & a \arrow[d, mapsto] \\ B \arrow[r] & A \otimes_R B & a \otimes 1 \\ b \arrow[r, mapsto] & 1 \otimes b & \end{tikzcd}$$

(Check this commutes!)

Suppose Z is a scheme with $Z \to X$ and $Z \to Y$ compatible with S-scheme structure. Then we get induced maps on global sections:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & A \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & A \otimes_R B \arrow[dr, dashed] & \\
& & \mathcal{O}_Z(Z)
\end{tikzcd}$$

The universal property of tensor product says there is a unique homomorphism $A \otimes_R B \to \mathcal{O}_Z(Z)$ making the diagram commute.

> Exercise: Morphisms to affine schemes $Z \to \operatorname{Spec} R$ are in one-to-one correspondence w/ maps on global sections $R \to \mathcal{O}_Z(Z)$.

Thus, there is a unique morphism $Z \to \operatorname{Spec}(A \otimes_R B)$ making the corresponding diagram of schemes commute. $\square$

In the general case, one can prove existence by first giving affine covers and then glueing (see pf of Thm 3.3 in Hartshorne – Make sure you read it!!)

Ex: If $A = k[x_1, \dots, x_n]$, $B = k[y_1, \dots, y_m]$, then $A \otimes_k B \cong k[x_1, \dots, x_n, y_1, \dots, y_m]$. Thus, $\mathbb{A}^n \times_k \mathbb{A}^m \cong \mathbb{A}^{m+n}_k$

Ex: If R is a k-algebra, then

$$R \otimes_k k[t] \cong R[t]$$

So $\mathbb{A}^1_k \times_k \operatorname{Spec} R \cong \mathbb{A}^1_R$.

This is an example of "base change." More generally, if X is an S-scheme and S' is also an S-scheme, then the scheme $X' = X \times_S S'$ is an S'-scheme. We say X' is obtained from X by making the base change or base extension $S' \to S$. (often the base schemes are Spec of two different fields)

Ex: let $A = \mathbb{R}[x,y]/(x^2+y^2)$. $\operatorname{Spec} A$ is irreducible, since (x^2+y^2) is prime. Then

$$\operatorname{Spec} A \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C} \cong \operatorname{Spec} \mathbb{C}[x,y]/(x^2+y^2)$$

is $\operatorname{Spec} A$ "base changed" from $\mathbb{R} \to \mathbb{C}$, and it is now reducible!

Fibers of morphisms

Another use of the fiber product is for describing the fibers of a morphism.

Let $f: X \to Y$ be a morphism of schemes and $P \in Y$ a point. Then P is a Y-scheme via the canonical map $k(P) \to Y$.

Def: The <u>scheme-theoretic fiber of f over P</u> is the scheme $X_P := X \times_Y \operatorname{Spec} k(P)$.

Claim The scheme theoretic fiber X_P is homeomorphic to the set-theoretic fiber over P, $f^{-1}(P)$ (given the subspace topology in X) via the "projection"
$X \times_Y \operatorname{Spec} k(P) \to X$.

Pf: First note that if $V \subseteq Y$ is an open neighborhood of P, then we can replace Y with V and thus we can assume Y is affine. i.e. $Y = \operatorname{Spec} A$.

Let $\{U_i\}$ be an open affine cover of X. Then it suffices to show that

$$U_i \times_Y \operatorname{Spec} k(P) \longrightarrow U_i$$

is a homeomorphism onto its image $U_i \cap f^{-1}(P)$. If $U_i = \operatorname{Spec} B$, this is equivalent to the following commutative algebra fact.

If $A \to B$ is a map of rings, then the fiber over $P \in \operatorname{Spec} A$ in the induced Spec map is homeomorphic to $\operatorname{Spec}(B \otimes_A k(P))$. (We proved this in 523.) $\square$

This construction allows us to view a morphism as a family of schemes, i.e. the fibers, parametrized by points in the image.

Def: If X_0 is a k-scheme, then a <u>family of deformations of X_0</u> is a morphism $f: X \longrightarrow Y$ s.t. for some $y_0 \in Y$, $k(y_0) = k$ and $X_{y_0} \cong X_0$. The other fibers are deformations of X_0.

Ex: Consider the map $\operatorname{Spec} \mathbb{Z}[x,y]/(x^2+y^2) \to \operatorname{Spec} \mathbb{Z}$.

This has fibers over closed points:

$$\operatorname{Spec} \left(\mathbb{Z}[x,y]/(x^2+y^2) \otimes_{\mathbb{Z}} \mathbb{F}_p \right)$$

$$\cong \operatorname{Spec} \mathbb{F}_p[x,y]/(x^2+y^2)$$

and generic fiber (fiber over (0))

$$\operatorname{Spec} \mathbb{Q}[x,y]/(x^2+y^2)$$

For properties which are open in families (e.g. smoothness) if any member in the family has it, the generic member will, since the generic point is in <u>every</u> nonempty open set (assuming irred. base)

This leads to "reduction to characteristic p": To study $\operatorname{Spec} \mathbb{Q}[x,y]/(x^2+y^2)$, we instead study $\operatorname{Spec} \mathbb{F}_p[x,y]/(x^2+y^2)$ for "typical p".

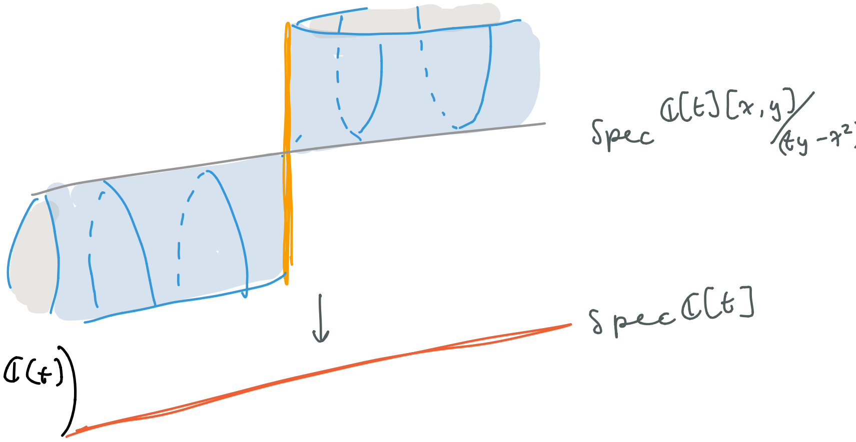
Note that 2 is not typical! The fiber over (2) is nonreduced. In fact, being reduced is an open condition.

Ex: Consider the family $\mathbb{C}[t][x,y]/(ty-x^2) \to \mathbb{C}[t]$.

The fiber over the closed point $(t-a)$ is

$$\operatorname{Spec}\left(\mathbb{C}[t][x,y]/(ty-x^2) \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/(t-a) \right)$$

$\cong \operatorname{Spec} \mathbb{C}[x,y]/(ay-x^2)$, which is a parabola in the general case, and a "double line" $\operatorname{Spec} \mathbb{C}[x,y]/(x^2)$ when $a=0$.



Note that the generic fiber is $\operatorname{Spec}\left(\mathbb{C}[t][x,y]/(ty-x^2) \otimes_{\mathbb{C}[t]} \mathbb{C}(t) \right)$

$\cong \operatorname{Spec}\left(\mathbb{C}(t)[x,y]/(ty-x^2) \right)$, which is itself a parabola over the field $\mathbb{C}(t)$. i.e. it looks like a "typical" closed fiber.